数理解析研究所講究録
292巻 1977年 98-126

COMMUTATIVE SUBARCHIMEDEAN SEMIGROUPS

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ABSTRACT

A commutative semigroup is called subarchimedean if it has an archimedean component which is an ideal. A commutative subarchimedean semigroup S without idempotent (called SAIF-semigroup) is called the first kind if the greatest cancellative homomorphic image of S is an $\mathfrak{N}$-semigroup. Every SAIF-semigroup S contains a subarchimedean maximal cancellative subsemigroup M and the structure of S is observed as the extension of M.

1. Introduction

A commutative semigroup S is called archimedean if, for every a, b $\in$ S, there is a positive integer m and an element c $\in$ S such that $a^m = bc$. As generalization of archimedeaness, the concept of subarchimedeaness is defined as stated in the abstract. A commutative cancellative archimedean (subarchimedean) semigroup without idempotent is called an $\mathfrak{N}$- ($\overline{\mathfrak{N}}$-) semigroup. The concept of subarchimedeaness plays an important role in the study of archimedean semigroups. For example, the translation semigroup of an $\mathfrak{N}$-semigroup is an $\overline{\mathfrak{N}}$-semigroup; a commutative archimedean semigroup without idempotent contains a maximal cancellative subsemigroup which is subarchimedian. In this paper we will deal with the last example in more general case, that is, we will study commutative subarchimedean semigroups as an extension of a maximal caneellative subsemigroup.

In Section 2 we will give the basic results of commutative subarchimedean semigroups. Subarchimedeaness is characterized in terms of the archimedean ideal or some homomorphic images. Further we will point out that there are two types called the first kind and the second kind. This paper deals with only the first kind. In Section 3 it will be proved that, in any commutative subarchimedean semigroup without idempotent, there is a maximal cancellative subsemigroup which is an $\overline{\mathfrak{N}}$-semigroup. In Section 4, we will describe the structure as the extension of an $\overline{\mathfrak{N}}$-semigroup; we call it "branch-growth." The concepts of essential semigroups and twigy semigroups appear at the step of extensions. Finally the two examples will show that the two concepts, essentiality and twiginess, are independent. It will turn out that "branch-growth" is a refinement of Putcha's concept of mild-ideal. However there remain many questions in constructing "branch-growth." Nevertheless, it is hoped that this paper will give a foundation for the future study. With respect to the study of commutative archimedean semigroups without idempotent from another point of view, see [4] [6] [8] [10] and so on.

2. Subarchimedean Semigroups

Throughout this paper, Z denotes the set of integers, Z_+ (Z_-) the set of positive (negative) integers, Z_+^o (Z_-^o) the set of non-negative (non-positive) integers. The operation is the usual addition.

<u>Definition</u> 2.1. A commutative semigroup S is called <u>subarchimedean</u> if there is an element $p \in S$ satisfying the following condition:

(2.1.1) For each $x \in S$, there is an $m \in Z_+$ and $y \in S$ such that $p^m = xy$.

The element p is called a <u>pivot element</u> of S and the set of p's is called the <u>pivot</u> of S, and denoted by $\mathrm{Piv}(s)$. If S is a commutative semigroup, $\mathrm{Piv}(S) \neq \emptyset$ if and only if S is subarchimedean; S is archimedean if and only if $\mathrm{Piv}(S) = S$.

In this paper only commutative semigroups are treated, so we will omit the word "commutative." For simplicity we use the following abbreviation.

<u>Definition</u> 2.2.

IF-semigroup is an idempotent-free semigroup, i.e., a semigroup without idempotent.

CIF-semigroup is a cancellative idempotent-free semigroup.

$\mathfrak{N}$-semigroup is a cancellative archimedean idempotent-free semigroup, i.e., CAIF-semigroup.

CSA-semigroup is a cancellative subarchimedean semigroup.

CSAI-semigroup is a cancellative subarchimedean semigroup with idempotent.

AIF-semigroup is an archimedean idempotent-free semigroup.

SAIF-semigroup is a subarchimedean idempotent-free semigroup.

$\overline{\mathfrak{N}}$-semigroup is a cancellative subarchimedean idempotent-free semigroup, i.e., CSAIF-semigroup. (See $\overline{\mathfrak{N}}$-semigroup in [3].)

The following are the fundamental properties of SAIF-semigroups. These properties are the natural extension of the properties of archimedean semigroups.

<u>Proposition</u> 2.3

(2.3.1) <u>If</u> S <u>is a subarchimedean semigroup and if</u> S <u>is homomorphic onto</u> S', <u>then</u> S' <u>is a subarchimedean semigroup</u>.

(2.3.2) <u>If</u> $S_1,\ldots,S_n$ <u>are subarchimedean semigroups, then</u> $S_1 \times \cdots \times S_n$ <u>is a subarchimedean semigroup</u>.

(2.3.3) <u>If</u> S <u>is a subarchimedean semigroup, then an ideal of</u> S <u>is subarchimedean</u>.

Proof. (2.3.1) Let $h: S \to S'$ be a surjective homomorphism. We show that if $p \in Piv(S)$, $h(p) \in Piv(S')$. Let $x' \in S'$. As h is onto, $x' = h(x)$ for some $x \in S$. By assumption, there is $m \in Z_+$ and $y \in S$ such that $p^m = xy$. Then $h(p)^m = x'\cdot h(y)$, hence $h(p) \in Piv(S')$.

(2.3.2) We need to prove only the case $n = 2$. Then the conclusion will be obtained by induction on n. Let $p \in Piv(S_1)$, $q \in Piv(S_2)$. We will show $(p,q) \in Piv(S_1 X S_2)$. Let $(x,y) \in S_1 \times S_2$. By assumption $p^m = xz$, $q^n = yu$ for some m, $n \in Z_+$, $m > 1$, $n > 1$, and some $z \in S_1$, $u \in S_2$. Then

$$(p,q)^{mn} = (p^{mn},q^{mn}) = (x^n z^n, y^m u^m) = (x,y)(x^{n-1}z^n, y^{m-1}u^m).$$

(2.3.3) Let I be an ideal of S. Let $p \in Piv(S)$ and $b \in I$. We will show $pb \in Piv(I)$. Let $x \in I$. By assumption, $p^m = xy$ for some $y \in S$ and $m \in Z_+$. Then $pb \in I$ and $(pb)^m = x(yb^m)$ where $yb^m \in I$.

Proposition 2.3 can be stated in terms of the notation "Piv" as follows:

<u>Proposition</u> 2.3'.

(2.3.1') <u>If</u> h <u>is a homomorphism</u>, $h(Piv(S)) \subseteq Piv(h(S))$.

(2.3.2') $Piv(S_1) \times \ldots \times Piv(S_n) \subseteq Piv(S_1 \times \ldots \times S_n)$.

(2.3.3') <u>If</u> I <u>is an ideal of</u> S, $I\cdot Piv(S) \subseteq Piv(I)$. <u>In addition</u>,

(2.3.4) <u>If</u> I <u>is an ideal of</u> S, $Piv(I) \subseteq Piv(S)$.

(2.3.5) $S\cdot Piv(S) \subseteq Piv(S)$, <u>that is</u>, $Piv(S)$ <u>is an ideal of</u> S.

(2.3.6) $\mathrm{Piv}(S) = \mathrm{Piv}(\mathrm{Piv}(S))$, _that is_, $\mathrm{Piv}(S)$ _is archimedean_.

Proof. (2.3.4) Let $p \in \mathrm{Piv}(I)$ and $x \in S$. Then $xp \in I$ as $p \in I$. By assumption there is $m \in Z_+$ and $y \in I$ such that $p^m = (xp)y = x(py)$. Hence $P \in \mathrm{Piv}(S)$.

(2.3.5) If $p \in \mathrm{Piv}(S)$ and $x \in S$, $p^m = xy$ for some $y \in S$, and $m \in Z_+$. Then, if $b \in S$, $(pb)^m = x(yb^m)$. This shows $pb \in \mathrm{Piv}(S)$ for all $b \in S$.

(2.3.6) Let $p, q \in \mathrm{Piv}(S)$. By definition $p^m = qx$ for some $x \in S$ and $m \in Z_+$. Then $p^{m+1} = q(xp)$, but $xp \in \mathrm{Piv}(S)$ by (2.3.5). Thus $p \in \mathrm{Piv}(\mathrm{Piv}(S))$, hence $\mathrm{Piv}(S) \subseteqq \mathrm{Piv}(\mathrm{Piv}(S))$. Since $\mathrm{Piv}(S)$ is an ideal of S by (2.3.5),

$\mathrm{Piv}(S) \subseteqq \mathrm{Piv}(\mathrm{Piv}(S)) \subseteqq \mathrm{Piv}(S)$ by (2.3.4),

hence $\mathrm{Piv}(S) = \mathrm{Piv}(\mathrm{Piv}(S))$.

Proposition 2.4. _If_ S _is subarchimedean, then_ $\mathrm{Piv}(S)$ _equals the archimedean component of_ S _which is an ideal of_ S.

Proof. Let $a \in \mathrm{Piv}(S)$ and S_0 be the archimedean component containing a. Then $\mathrm{Piv}(S) \subseteqq S_0$ because $\mathrm{Piv}(S)$ is archimedean by (2.3.6). As $\mathrm{Piv}(S)$ is an ideal of S by (2.3.5), for any $x \in S$, we have $xa \in \mathrm{Piv}(S) \subseteqq S_0$. Let $b \in S_0$. Since S_0 is archimedean, $b^n = (xa)c = x(ac)$ for some $c \in S_0$ and some $n \in Z_+$. Hence $S_0 \subseteqq \mathrm{Piv}(S)$. Thus $S_0 = \mathrm{Piv}(S)$.

Definition 2.5. Let S be a commutative semigroup and let $p \in S$. Define ρ_p and η_p by

$x \, \rho_p \, y$ if and only if $p^m x = p^n y$ for some $m, n \in Z_+$.

$x \, \eta_p \, y$ if and only if $p^m x = p^m y$ for some $m \in Z_+$.

(2.6) _Both_ ρ_p _and_ η_p _are congruences on_ S _for all_ $p \in S$.

Let $\mathbb{R}_+^o$ ($\mathbb{R}_+$) be the semigroup of non-negative (positive) real numbers under addition.

<u>Proposition</u> 2.7. <u>Let</u> S <u>be either an</u> SAIF-<u>semigroup or a</u> CSAI-<u>semigroup which is not a group</u>.

(2.7.1) <u>If</u> $p \in \mathrm{Piv}(S)$, <u>then</u> $px \neq x$ <u>for all</u> $x \in S$.

(2.7.2) η_p <u>is the smallest cancellative congruence on</u> S <u>for all</u> $p \in \mathrm{Piv}(S)$.

(2.7.3) <u>There is</u> $f \in \mathrm{Hom}(S, \mathbb{R}_+^o)$ <u>such that</u> $f(x) > 0$ <u>for all</u> $x \in \mathrm{Piv}(S)$.

(2.7.4) $\bigcap_{n=1}^{\infty} p^n S = \emptyset$ <u>for all</u> $p \in \mathrm{Piv}(S)$.

Proof. From the assumption it follows that Piv(S) is an AIR-semigroup.

(2.7.1) If $x \in \mathrm{Piv}(S)$, $px \neq x$ by [10]. If $x \notin \mathrm{Piv}(S)$, then $px \in \mathrm{Piv}(S)$ since Piv(S) is an ideal. Hence $px \neq x$.

(2.7.2) By (2.6) η_p is a congruence. Assume $xy \; \eta_p \; xz$. By definition $p^m xy = p^m xz$ for some $m \in Z_+$; but $p^n = ux$ for some $u \in S$ and some $n \in Z_+$. From $p^m xy = p^m xz$ it follows that $p^{m+n} y = p^{m+n} z$, hence $y \; \eta_p \; z$. Thus η_p is a cancellative congruence on S. Let η' be a cancellative congruence on S. If $x \; \eta_p \; y$, then $p^m x = p^m y$ for some $m \in Z_+$, but $p^m x \; \eta' \; p^m y$ because η' is reflexive. Then we get $x \; \eta' \; y$ since η' is cancellative. Thus $\eta_p \subseteq \eta'$. Hence η_p is the smallest cancellative congruence on S.

(2.7.3) Let $S_0 = \mathrm{Piv}(S)$. Since S_0 is an AIF-semigroup, there is a non-trivial $f \in \mathrm{Hom}(S_0, \mathbb{R}_+)$. This is due to [5], [12]. It is sufficient to prove that f can be extended to $\bar{f} \in \mathrm{Hom}(S, \mathbb{R}_+^o)$. Define $\bar{f}$ by $\bar{f}(x) = f(ax) - f(a)$ where $a \in S_0$. It is easy to see that $\bar{f}$ is a well defined homomorphism and an extension of f. We need only to show $\bar{f}(x) \geqq 0$. Suppose $\bar{f}(x_1) < 0$ for some $x_1 \in S$. Choose $m \in Z_+$ such that $\bar{f}(a) + m\bar{f}(x_1) < 0$. Then $\bar{f}(ax_1^{\,m}) < 0$, but $\bar{f}(ax_1^{\,m}) \geqq 0$ since $ax_1^{\,m} \in S_0$. We arrive at contradiction. Hence $\bar{f} \in \mathrm{Hom}(S, \mathbb{R}_+^o)$.

(2.7.4) By (2.7.3) there is an $f \in \mathrm{Hom}(S, \mathbb{R}_+^o)$ with $f(p) > 0$, $p \in \mathrm{Piv}(S)$. Suppose $\bigcap_{n=1}^{\infty} p^n S \neq \emptyset$. There is a $y \in S$ such that $y = p^n x_n$ for some $x_n \in S$ $(n=1,2,\ldots)$. Then $f(y) \geqq f(p^n) = nf(p) > 0$ for all $n \in Z_+$. This is, however, a contradiction, because there is an $n \in Z_+$ such that $f(p^n) = nf(p) > f(y)$. Therefore we have proved the claim.

According to (2.7.2), $\eta_p = \eta_q$ for all $p, q \in \mathrm{Piv}(S)$. So, when S is subarchimedean, let $\eta = \eta_p$.

The following theorem characterizes subarchimedeaness in IF-semigroups.

Theorem 2.8. *Let S be an IF-semigroup. The following are equivalent.*

(2.8.1) *S is subarchimedean.*

(2.8.2) *S has an ideal which is an archimedean component.*

(2.8.3) *S/ρ_a is a group for some $a \in S$.*

(2.8.4) *S/ρ_a is a CSA-semigroup for some $a \in S$.*

(2.8.5) *S/ρ_a is subarchimedean for some $a \in S$.*

(2.8.6) *S/ρ_a is subarchimedean for all $a \in S$.*

(2.8.7) *S/η_a is a CSA-semigroup for some $a \in S$.*

Proof. (2.8.1) => (2.8.2): This is done by Proposition 2.4.

(2.8.2) => (2.8.3): Let I be the ideal which is an archimedean component, and let $a \in I$. Then we will show that S/ρ_a is a group. It is known in [1], [7], [9] that $I(\rho_a|I)$ is a group. For every $x \in S$, $x \,\rho_a\, ax$ where $ax \in I$. It follows that $S/\rho_a \cong I/(\rho_a|I)$, hence S/ρ_a is a group.

(2.8.3) => (2.8.4) and (2.8.4) => (2.8.5) are obvious.

(2.8.5) => (2.8.1) [3]: Let $g: S \to S/\rho_a$ be the natural homomorphism and let $g(x) = \bar{x}$. Let $\bar{v} \in \mathrm{Piv}(S/\rho_a)$. For all $\bar{x} \in S/\rho_a$, there exists an $m \in Z_+$ and $\bar{y} \in S/\rho_a$ such that $\bar{v}^m = \bar{x}\,\bar{y}$.

By definition of ρ_a, $v^m a^k = xya^\ell$ for some $k, \ell \in Z_+$. Then $(va)^{m+k} = x(ya^{\ell+m}v^k)$ where x is an arbitrary element of S; so we see $va \in Piv(S)$.

So far we have seen that the first five conditions are equivalent.

(2.8.1) => (2.8.6): This is obtained by Proposition 2.3, (2.3.1).

(2.8.6) => (2.8.5): Obvious.

(2.8.1) => (2.8.7): See Proposition 2.7, (2.7.2).

(2.8.7) => (2.8.5): Obviously $\eta_a \subseteq \rho_a$, hence S/η_a is homomorphic to S/ρ_a. Since S/η_a is subarchimedean, S/ρ_a is also subarchimedean by Proposition 2.3, (2.3.1). This completes the proof.

Let S be an IF-semigroup and define $Gr(S)$ by

$$Gr(S) = \{a \in S: S/\rho_a \text{ is a group}\}.$$

Remark 2.9.1. $Piv(S) = Gr(S)$.

Proof. Obviously $Piv(S) \subseteq Gr(S)$. Let $a \in Gr(S)$. As S/ρ_a is a group, for every $x \in S$, there is a $y \in S$ such that $xy \; \rho_a \; a$ namely $a^m = x(ya^n)$ for some $m, n \in Z_+$. This shows $a \in Piv(S)$, hence $Gr(S) \subseteq Piv(S)$.

Corollary 2.9.2 $Gr(S) = S$ if and only if S is archimedean.

Remark 2.9.3 Define $Ca(S) = \{a \in S: S/\rho_a \text{ is a CSA-semigroup}\}$. Then $Gr(S) \subseteq Ca(S)$ but $Ca(S) \neq Gr(S)$ in general.

Proof. The first part is evident. We show a counterexample for the second. Let S_0 be an $\mathfrak{N}$-semigroup and $S_1 = Z_+$. Let $S = S_0 \cup S_1$. If $x \in S_0$ and $n \in Z_+$, define $x \cdot n$ by

$$x \cdot n = n \cdot x = x \qquad \text{for all } x \in S_0,\ n \in Z_+.$$

Then S is an SAIF-semigroup and $1 \in Ca(S)$ but $1 \notin Gr(S)$.

When S is subarchimedean and $a \in \mathrm{Piv}(S)$, S/ρ_a is called the structure group of S with respect to a, and the element a is called a standard element of the structure group S/ρ_a.

Theorem 2.10. Let η denote the smallest cancellative congruence. If S is an SAIF-semigroup, S/η is either an $\overline{\mathfrak{N}}$-semigroup or a CSAI-semigroup which is not a group.

Proof. By Proposition 2.7, S/η is a CSA-semigroup. If S/η has no idempotent, S/η is an $\overline{\mathfrak{N}}$-semigroup. Suppose that S/η is a group. If S is archimedean, S must contain an idempotent by Corollary 3.4 of [11] and Theorem 3.3 of [11]. This is a contradiction. If S is not archimedean, S has a pivot V which is an AIF-semigroup and a proper ideal of S. By Proposition 3.1 of [11] and Corollary 3.4 of [11], V must contain an idempotent. This is again a contradiction. Therefore S/η is not a group.

Definition 2.11. Let η be the smallest cancellative congruence on S. An SAIF-semigroup S is called the first kind if S/η is an $\overline{\mathfrak{N}}$-semigroup; S is called the second kind if S/η contains an idempotent.

Proposition 2.12. Let S be an SAIF-semigroup. The following are equivalent.

(2.12.1) S is of the second kind.

(2.12.2) For every $a \in \mathrm{Piv}(S)$, there is $x \in S$ and $m \in Z_+$ such that $a^m x = a^m$.

(2.12.3) $yx = y$ for some $x, y \in S$.

(2.12.4) $bx = b$ for some $b \in \mathrm{Piv}(S)$, some $x \in S$.

Proof. (2.12.1) => (2.12.2): Let $a \in \mathrm{Piv}(S)$. As S/η_a has an idempotent, $a^n x^2 = a^n x$ for some $x \in S$, some $n \in Z_+$. By subarchimedeaness, $a^\ell = xz$ for some $z \in S$, some $\ell \in Z_+$; hence we get $a^m x = a^m$ where $m = n + \ell$.

(2.12.2) => (2.12.3): Obvious.

(2.12.3) => (2.12.4): Let $a \in \mathrm{Piv}(S)$. Then $yx = y$ implies $(ay)x = ay$ where $ay \in \mathrm{Piv}(S)$ since $\mathrm{Piv}(S)$ is an ideal.

(2.12.4) => (2.12.1): $bx = b$ implies $bx^2 = bx$, hence $x^2 \eta x$.

Accordingly the negation of each of (2.12.2), (2.12.3) and (2.12.4) is a necessary and sufficient condition for S to be of the first kind. In this paper we discuss only SAIF-semigroups of the first kind.

Corollary 2.13. *Let S be a commutative semigroup. S is an SAIF-semigroup of the first kind if and only if S is subarchimedean and there is an $a \in \mathrm{Piv}(S)$ such that $a^m x \neq a^m$ for all $m \in Z_+$ and all $x \in S$.*

Example 2.14. An AIF-semigroup is an SAIF-semigroup of the first kind.

Example 2.15. Let $S_0 = \{1,2,3,\ldots\}$ and $S_1 = \{2',3',4',\ldots\}$ be the additive semigroups where $x' + y' = (x+y)'$ in S_1. Let $S = S_0 \cup S_1$ and define a binary operation in S such that S_0 is an ideal of S, and S_1 is a sub-semigroup of S, and if $a \in S_0$ and $b' \in S_1$,

$$a + b' = b' + a = a + b \qquad \text{for all } a \in S_0,\ b' \in S_1.$$

Then S is an SAIF-semigroup of the first kind. If we define

$$a + b' = b' + a = a \qquad \text{for all } a \in S_0,\ b' \in S_1,$$

then S is an SAIF-semigroup of the second kind.

More complicated examples will be seen at the end of Section 3.

3. Existence of Subarchimedean Maximal Cancellative Subsemigroups

Let S be an SAIF-semigroup and let $a \in S$. Let $\mathfrak{M}_a$ be the set of all cancellative subsemigroups of S containing a fixed a. $\mathfrak{M}_a \neq \emptyset$ since $\mathfrak{M}_a$ contains the cyclic subsemigroup generated by a. If $\{M_\xi : \xi \in \Xi\}$ is a chain of subsets of $\mathfrak{M}_a$, i.e., $M_{\xi_1} \subset M_{\xi_2}$, $\xi_1 < \xi_2$, then we see that the union $\bigcup_{\xi \in \Xi} M_\xi \in \mathfrak{M}_a$. By Zorn's lemma, $\mathfrak{M}_a$ has a maximal element.

Proposition 3.1. Let S be an SAIF-semigroup and $\varphi: S \to S/\eta$ be the greatest cancellative homomorphism. Let $a \in \mathrm{Piv}(S)$ and C a cancellative subsemigroup of S containing a. Then there is an isomorphism Ψ of C into S/η such that $\Psi = \varphi\iota$ where ι is the inclusion map of C into S.

Proof. Define Ψ by $\Psi(x) = \varphi(x)$ for $x \in C$. (We denote $\varphi\iota(x)$ by $\varphi(x)$.) We need only to show that Ψ is one-to-one. Assume $x, y \in C$ and $\Psi(x) = \Psi(y)$. Then $\varphi(x) = \varphi(y)$ implies $a^m x = a^m y$ for some $m \in Z_+$. By cancellation in C, we have $x = y$.

Let $a \in \mathrm{Piv}(S)$, $G = S/\rho_a$ and let $g: S \to G$ be the natural homomorphism: $g(S) = G$. Assume $S = \bigcup_{\lambda \in G} S_\lambda$ is the decomposition of S due to g. Notationally $g(x) = \lambda$ if and only if $x \in S_\lambda$.

Theorem 3.2. Let S be an SAIF-semigroup. For each $a \in \mathrm{Piv}(S)$, there exists a maximal cancellative subsemigroup M of S containing a (that is, M is maximal in $\mathfrak{M}_a$) such that $g(M) = G$. The M is necessarily an $\overline{\mathfrak{N}}$-semigroup.

Proof. Let $\mathfrak{G}_a$ be the set of all cancellative subsemigroups T of S containing a having property that $g(T)$ is a subgroup of G. $\mathfrak{G}_a \neq \emptyset$ since $g([a]) = \{\varepsilon\}$ where ε is the identity of G. By Zorn's lemma, we can see that there is a maximal element in $\mathfrak{G}_a$. Let M be a maximal element in $\mathfrak{G}_a$ and let $g(M) = H$. We will prove $H = G$. Suppose $H \neq G$, let $\alpha \in G \setminus H$.

Case I: $\alpha^m \notin H$ for all $m \in Z_+$.

Take arbitrarily $p \in S_\alpha$ and then choose $q \in S$ such that $pq = a^{s_0}$ where $s_0 \in Z_+$ and s_0 is the minimum. Of course $g(p) = \alpha$, $g(q) = \alpha^{-1}$ and $g(a) = \epsilon$ (the identity of G). Let M_1 be the sub-semigroup of S generated by M, p and q. As $pq = a^{s_0} \in M$, it is easily seen that every element of M_1 can be expressed as exactly one of the following:

$$x, \quad yp^m, \quad zq^n$$

where $x \in M$, $y, z \in M^1$ and $m, n \in Z_+$.

Moreover the uniqueness of expression is shown as follows. First of all we note that the following equalities are impossible:

$$x = yp^m, \quad x = zq^n, \quad yp^m = zq^n \quad (x, y, z \in M).$$

Suppose $x = yp^m$. Then $g(x) = g(y)\cdot\alpha^m$, hence $\alpha^m \in H$. If $x = zq^n$, $g(x) = g(z)\cdot\alpha^{-n}$, so $\alpha^{-n} \in H$. If $yp^m = zq^n$, then $g(y)\cdot\alpha^m = g(z)\cdot\alpha^{-n}$, hence $\alpha^{m+n} \in H$. In each case we arrive at a contradiction to the assumption on α.

Now assume $yp^m = zp^n$, $y\ z \in M^1$, and suppose $m \neq n$, say $m > n$. Then $g(y)\cdot\alpha^m = g(z)\cdot\alpha^n$ which implies $\alpha^{m-n} \in H$, a contradiction, hence $m = n$. By multiplying the both sides of $yp^m = zp^n$ by q^m, we have $ya^{ms_0} = za^{ms_0}$, so $y = z$ follows by cancellation of M. Thus we have proved that $yp^m = zp^n$ implies $y = z$ and $m = n$. Similarly we have $yq^m = zq^n$, $y, z \in M^1$, implies $y = z$ and $m = n$. Thus the uniqueness of expression has been proved. By using this we prove cancellation of M_1. Since the equality $x(yp^m) = x(zq^n)$ does not occur, we start with $x(yp^m) = x(zp^n)$. Then $xy = xz$ and $m = n$ by uniqueness; now $y = z$ by cancellation of M. Similarly $x(yq^m) = x(zq^n)$ implies $y = z$ and $m = n$. Summarizing the above together with cancellation of M, we have

$$xb = xc, \quad x \in M,\ b, c \in M_1, \text{ implies } b = c.$$

To consider the remaining case, suppose $(xp^m)b = (xp^m)c$ where

$b, c \in M_1$. Multiplying the both sides by q^m, we get $(xa^{ms_0})b = (xa^{ms_0})c$ where $xa^{ms_0} \in M$. By the preceding result, $b = c$. Similarly $(xq^m)b = (xq^m)c$ implies $b = c$. Thus we have shown cancellation of M_1. Certainly $g(M_1) = H\cdot[\alpha]$ is a subgroup of G and $M \subsetneqq M_1$. This contradicts maximality of M in $\mathcal{G}_a$.

Case II: $a^m \in H$ for some $m \in Z$, $m \neq 0$.

Without loss of generality we can assume $m \in Z_+$. (If $m \in Z_-$, consider α^{-1} instead of α.) Let m_0 be the minimum of positive m's with $\alpha^m \in H$. As $\alpha \notin H$, $m_0 > 1$. Let $\beta = \alpha^{m_0}$ and choose arbitrarily $p \in S_\alpha$. Take $x_\beta \in S_\beta \cap M$, then $a^s p^{m_0} = a^t x_\beta$ for some $s, t \in Z_+$. If $m_0 \geqq s$, then $(ap)^{m_0} = a^{m_0-s+t} x_\beta$. If $m_0 < s$, then we can find $k \in Z_+$ and $r \in Z_+^o$ such that $m_0 k = s + r$, $0 \leqq r < m_0$; then $(a^k p)^{m_0} = a^{t+r} x_\beta$. Let $q = ap$ in the first case; let $q = a^k p$ in the second case. Then $q \in S_\alpha$, $q \notin M$ but $q^{m_0} \in M$ where m_0 is the minimum. Let M_1 be the subsemigroup of S generated by M and q. Every element of M_1 has a unique expression of the form, xq^m where $x \in M^1$, $0 \leqq m < m_0$, but if $m = 0$, then $x \in M$, and xq^o denotes x itself. The uniqueness of expression is shown as follows. Let $xq^m = yq^n$, $x, y \in M^1$, $0 \leqq m < m_0$, $0 \leqq n < m_0$. We can assume that at least one of m and n is in Z_+. Now suppose $m \neq n$, say $m > n$. Then $g(x)\cdot\alpha^m = g(y)\cdot\alpha^n$ implies $\alpha^{m-n} \in H$ where $0 < m-n < m_0$. This contradicts minimality of m_0. Therefore $m = n$. Since S is subarchimedean, there is $u \in S$ and $t \in Z_+$ such that $a^t = q^m u$. Now $xq^m = yq^m$ implies $xa^t = ya^t$, and then we get $x = y$ by cancellation of M. Thus the uniqueness has been shown. To show cancellation of M_1, assume

$$(xq^m)(yq^n) = (xq^m)(zq^\ell), \quad x, y, z \in M^1$$

or

$$(3.2.1) \quad xyq^{m+n} = xzq^{m+\ell}.$$

Here we assume $0 \leqq m < m_0$, $0 \leqq n < m_0$, $0 \leqq \ell < m_0$; then $0 \leqq m+n < 2m_0$, $0 \leqq m+\ell < 2m_0$.

Let $m + n = im_0 + r$, $0 \leqq r < m_0$, $i=0$ or 1

$m + \ell = jm_0 + s$, $0 \leqq s < m_0$, $j=0$ or 1.

Then (3.2.1) becomes $(xyq^{im_0})q^r = (xzq^{jm_0})q^s$. By the uniqueness, we have $r = s$ and

$$(3.2.2) \quad xyq^{im_0} = xzq^{jm_0};$$

then $m + n \equiv m + \ell \pmod{m_0}$ implies $n \equiv \ell \pmod{m_0}$, but since $0 \leqq n < m_0$ and $0 \leqq \ell < m_0$, we get $n = \ell$; accordingly $m + n = m + \ell$ implies $i = j$. Finally $y = z$ follows from (3.2.2) by cancellation of M. Hence M_1 is cancellative, and $g(M_1)$ is the subsemigroup of G generated by H and α where $\alpha^{m_0} \in H$. It is easy to see that $g(M_1)$ is a subgroup of G and $M \subsetneqq M_1$; we arrive at contradiction to the maximality of M in $\mathfrak{G}_a$.

In both Case I and Case II we have shown that $H = g(M) = G$. Let $\mathfrak{M}_M$ be the set of all cancellative subsemigroups of S containing a fixed maximal element M of $\mathfrak{G}_a$ which was obtained above. By Zorn's lemma, $\mathfrak{M}_M$ has a maximal element M_0. Obviously $\mathfrak{M}_M \subset \mathfrak{M}_a$ where $\mathfrak{M}_a$ was defined at the beginning of this section. Now M_0 is also maximal in $\mathfrak{M}_a$, and $M \subseteqq M_0$ implies $G = g(M) \subseteqq g(M_0)$, hence $g(M_0) = G$. Thus $M_0 \in \mathfrak{G}_a$ and so $M = M_0$ by maximality of M. By Theorem 2.8 and the assumption on S, we conclude M is maximal in $\mathfrak{M}_a$, $g(M) = G$ and M is an $\overline{\mathfrak{N}}$-semigroup.

<u>Corollary</u> 3.3. <u>Let</u> S <u>be</u> <u>an</u> SAIF-<u>semigroup</u> <u>and</u> $a \in \mathrm{Piv}(S)$. <u>Let</u> $g\colon S \to S/\rho_a = G$. <u>If</u> M <u>is</u> <u>maximal</u> <u>in</u> $\mathfrak{G}_a$ <u>then</u> M <u>is</u> <u>maximal</u> <u>in</u> $\mathfrak{M}_a$ <u>and</u> $g(M) = G$.

<u>Definition</u> 3.4. Let S be an SAIF-semigroup and $a \in \mathrm{Piv}(S)$. A subsemigroup M of S is called $\overline{\mathfrak{N}}$-<u>maximal</u> <u>containing</u> a if M is maximal in $\mathfrak{M}_a$ and $g(M) = G$, hence M is an $\overline{\mathfrak{N}}$-subsemigroup of S.

Therefore Theorem 3.2 proves the existence of $\overline{\mathfrak{N}}$-maximal subsemigroup which contains a pivot element. An $\overline{\mathfrak{N}}$-maximal subsemigroup is different from a maximal $\overline{\mathfrak{N}}$-subsemigroup in general. By Zorn's lemma we see the existence of a maximal $\overline{\mathfrak{N}}$- ($\mathfrak{N}$-) subsemigroup containing a pivot element.

Remark 3.5. Let S be an SAIF-semigroup. A maximal cancellative subsemigroup of S containing $a \notin \mathrm{Piv}(S)$ need not be an $\overline{\mathfrak{N}}$-semigroup as is shown in the following example:

Example 3.6. Let $1 < 2 < \ldots < n < \ldots < \omega$ and define S by

$$S_\omega = Z_+, \qquad S_n = \underbrace{Z_+ \times \cdots \times Z_+}_{n} \quad \text{for each } n \in Z_+;$$

$$S = \bigcup_{n=1}^{\infty} S_n \cup S_\omega,$$

and then define a binary operation in S as follows: If both X and Y are in the same S_i $(i=1,\ldots,n,\ldots,\omega)$, $X\cdot Y$ is already defined in S_i. If $(X_{m1},\ldots,x_{mm}) \in S_m$ and $(y_{n1},\ldots,y_{nn}) \in S_n$ and if $m \leqq n$ and $m=1,2,\ldots,$ then

$$(x_{m1},\ldots,x_{mm})\,(y_{n1},\ldots,y_{nn}) = (y_{n1},\ldots,y_{nn})\,(x_{m1},\ldots,x_{mm})$$
$$= (x_{m1}+y_{n1},\ldots,x_{mm}+y_{nm},\ y_{n,m+1},\ldots,y_{nn}).$$

Define $h\colon \bigcup_{i=1}^{\infty} S_i \to Z_+$ by $h(x_{m1},\ldots,x_{mm}) = x_{m1}$ $(m=1,2,\ldots)$. If $p \in S_\omega$ and $X \in \bigcup_{i=1}^{\infty} S_i$, define $p\cdot X = X\cdot p = p + h(x)$. Then S is an SAIF-semigroup of the first kind since $S/\eta \cong Z_+$. Let $x \in S$ and $M(x)$ be a maximal cancellative subsemigroup containing x. Then we have

$$M(x) = \begin{cases} S_\omega & \text{if } x \in S_\omega \\ \bigcup_{i=1}^{\infty} S_i & \text{if } x \in \bigcup_{i=1}^{\infty} S_i. \end{cases}$$

$M(x)$ is an $\mathfrak{N}$-semigroup if $x \in S_\omega$, but not an $\overline{\mathfrak{N}}$-semigroup if

$$x \in \bigcup_{i=1}^{\infty} S_i .$$

Instead of the above definition of p.X, if we define

$$p\cdot X = X\cdot p = p$$

then S is an SAIF-semigroup of the second kind since $S/\eta \cong Z_+^o$.
We have the same result on M(x).

Remark 3.7. Let S be an SAIF-semigroup of the first kind. Let $a \in \mathrm{Piv}(S)$. All maximal cancellative subsemigroups of S containing a are not necessarily $\overline{\mathfrak{N}}$-semigroups.

4. Branch-Growth

Following Putcha [8], a subsemigroup A of an SAIF-semigroup S is called a mild ideal of S if, for each $x \in S$, $xA \cap A \neq \emptyset$.

Fact 4.1. The following are fundamental properties of mild ideals.

(4.1.1) Mild ideals are preserved under homomorphisms.

(4.1.2) Let A, B and C be commutative semigroups such that $C \subset B \subset A$. If C is a mild ideal of A, then B is a mild ideal of A.

(4.1.3) If a subgroup H is a mild ideal of a group G, then H = G.

Proposition 4.2. ([8], [2]). Let S be an SAIF-semigroup, let $a \in \mathrm{Piv}(S)$. If M is maximal in $\mathfrak{M}_a$, then M is a mild ideal of S.

Definition 4.3. Let B be a subsemigroup of a commutative semigroup A, and J a subsemigroup of B. B is called a J-mild ideal of A if, for every $x \in A$, $xJ \cap B \neq \emptyset$.

Obviously every J-mild ideal of A is a mild ideal of A; conversely a mild ideal J of A is a J-mild ideal of A. The

M in Theorem 3.2 is an [a]-mild ideal of S where [a] is the cyclic subsemigroup of S generated by a.

Definition 4.4. Let A and B be commutative semigroups. A is called a branch-growth of B with respect to a if B is an [a]-mild ideal of A for some $a \in B$, and if

$$a^m x = a^n, \quad x \in A, \quad \text{implies } m < n.$$

We denote A by $\mathfrak{Br}_a(B)$. A branch-growth of B with respect to a is not uniquely determined by B and a, but $\mathfrak{Br}_a(B)$ denotes any one of those.

Fact 4.5. Let A, B, C and A_i $(i=1,2,\ldots)$ be commutative semigroups.

(4.5.1) If $C \subset B \subset A$ and if $A = \mathfrak{Br}_a(C)$, then $B = \mathfrak{Br}_a(C)$ and $A = \mathfrak{Br}_a(B)$.

(4.5.2) If $B = \mathfrak{Br}_a(C)$ and if $A = \mathfrak{Br}_a(B)$, then $A = \mathfrak{Br}_a(C)$.

(4.5.3) If $a \in A_1$ and if $A_{i+1} = \mathfrak{Br}_a(A_i)$ $(i \in Z_+)$, then

$$\bigcup_{i=1}^{\infty} A_i = \mathfrak{Br}_a(A_1).$$

Lemma 4.6. Let A, B be commutative semigroups. Let $a \in B \subseteqq A$, and $A = \mathfrak{Br}_a(B)$. Then A is an SAIF-semigroup of the first kind and $a \in \mathrm{Piv}(A)$ if and only if B is an SAIF-semigroup of the first kind and $a \in \mathrm{Piv}(B)$.

Proof. Necessity. Let $x \in B \subseteqq A$. Since A is subarchimedean, $a^m = xy$ for some $m \in Z_+$, some $y \in A$. Now $A = \mathfrak{Br}_a(B)$ implies $a^n y \in B$ for some $n \in Z_+$. Then $a^{m+n} = x(a^n y)$, hence B is subarchimedean and $a \in \mathrm{Piv}(B)$. Since A is of the first kind, $zu \neq z$ for all $z, u \in A$, hence for all $z, u \in B$; therefore B is of the first kind. We have used Proposition 2.12.

Sufficiency. Let $x \in A$. There is $\ell \in Z_+$ such that $a^\ell x \in B$ because $A = \mathfrak{Br}_a(B)$. Now B is subarchimedean: $a^k = (a^\ell x)z = x(a^\ell z)$ for some $k \in Z_+$ and some $z \in B$. Hence A is subarchimedean and $a \in \mathrm{Piv}(A)$. Suppose A is not of the first kind. There are $z, u \in A$ such that $zu = z$. However $a^m = zv$ for some $v \in A$, some $m \in Z_+$ because of subarchimedeaness. Then $zu = z$ implies $a^m u = a^m$, which is a contradiction to the assumption that $A = \mathfrak{Br}_a(B)$ (see Definition 4.4). Hence A is of the first kind.

Lemma 4.7. Let A_i $(i \in Z_+)$ be commutative semigroups. Let $a \in A_1 \subseteq A_2 \subseteq \dots$ such that $A_{i+1} = \mathfrak{Br}_a(A_i)$ $(i \in Z_+)$. $\bigcup_{i=1}^{\infty} A_i$ is an SAIF-semigroup of the first kind and $a \in \mathrm{Piv}(\bigcup_{i=1}^{\infty} A_i)$ if and only if each A_i is an SAIF-semigroup of the first kind and $a \in \mathrm{Piv}(A_i)$ $(i \in Z_+)$.

Lemma 4.8. Let A, B be SAIF-semigroups. Assume $a \in B \subseteq A$ and $A = \mathfrak{Br}_a(B)$. Then $\mathrm{Piv}(B) \subseteq \mathrm{Piv}(A)$.

Proof. Let $b \in \mathrm{Piv}(B)$ and let $x \in A$. Then $a^m x \in B$ for some $m \in Z_+$. By assumption, $b^n = (a^m x)y = x(a^m y)$ for some $y \in B$ and some $n \in Z_+$. Hence $b \in \mathrm{Piv}(A)$.

Let S be an SAIF-semigroup of the first kind, $a \in \mathrm{Piv}(S)$ and let M be an $\overline{\mathfrak{N}}$-maximal subsemigroup containing a. Define a sequence of subsemigroups of S as follows: "$y|z$ in M" means $z = yu$ for some $u \in M$.

$$(4.9) \quad \begin{cases} M_0 = M, \\ M_i = \{x \in S;\ a^m | a^i x \text{ in } M \text{ for some } m \geq i\} \quad (i \in Z_+), \\ M_\infty = \bigcup_{0 \leq i < \infty} M_i. \end{cases}$$

If $0 \leq i < j \leq \infty$, M_j is a branch-growth of M_i with respect to a; S is also a branch-growth of M_j with respect to a $(0 \leq j \leq \infty)$.

In addition, define

$$M_i^* = \{x \in S\colon a^m | a^i x \text{ in } M \text{ for some } m > i\} \quad (i \in Z_+).$$

Proposition 4.9. *M_λ and M_i^* are SAIF-subsemigroups of S of the first kind where $0 \leqq \lambda \leqq \infty$, $i \in Z_+$,*
$M \subseteqq M_1 \subseteqq \cdots \subseteqq M_i \subseteqq \cdots \subseteqq M_\infty$, and $M_{i+1} = \{x \in S\colon ax \in M_i^*\}$ where $i \in Z_+$.

Proof. It is easy to see that $M \subseteqq M_i$ for all $i \in Z_+$. Let $i, j \in Z_+$ with $i < j$. Then $a^i x = a^m b$, $i \leqq m$, implies $a^j x = a^{m+j-i} b$ where $j \leqq m + j - i$. Hence if $i < j$, $M_i \subseteqq M_j$. We show M_i is a subsemigroup for each $i \in Z_+$. Let $x, y \in M_i$, namely, $a^i x = a^m b$, $a^i y = a^n c$ where $i \leqq m$, $i \leqq n$ and $b, c \in M$. Then

$$a^i xy = a^m by = a^m yb = a^i y a^{m-i} b = a^{m+n-i} bc$$

where $i \leqq m + n - i$. Hence $xy \in M_i$.

Obviously $M_i \subseteqq M_\infty$ for all $i \in Z_+$ and M_∞ is a subsemigroup of S. When the equality is removed in the above proof, we see that M_i^* is a subsemigroup. It is easy to show $M_{i+1} = \{x \in S\colon ax \in M_i^*\}$; M_i $(i \in Z_+)$ is SAIF of the first kind by Lemma 4.6; M_∞ is SAIF-semigroup of the first kind by Lemma 4.7.

Specializing Proposition 3.1. We introduce the concept of "twigy."

Definition 4.10. An SAIF-semigroup S of the first kind is called *twigy* if there exists a cancellative subsemigroup M of S and an isomorphims ψ of M onto S/η such that $M \cap \mathrm{Piv}(S) \neq \emptyset$ and $\psi = \varphi\iota$ where η is the smallest cancellative congruence, φ is the natural homomorphism, and ι is the inclusion map.

$$\begin{tikzcd} S \arrow[rr, "\varphi"] & & S/\eta \\ & M \arrow[ul, "\iota"] \arrow[ur, "\psi"'] & \end{tikzcd}$$

Let $a \in M \cap \mathrm{Piv}(S)$. M is necessarily an $\overline{\mathfrak{N}}$-maximal subsemigroup of

S containing a. S is twigy if and only if there is an isomorphism Ψ of M into S/η such that $\varphi(S) \subseteqq \Psi(M)$.

Theorem 4.11. *Let S be an SAIF-semigroup of the first kind, and M be an $\overline{\mathfrak{N}}$-maximal subsemigroup of S containing a pivot element a of S. Then M_λ $(1 \leqq \lambda \leqq \infty)$ is a twigy subsemigroup whose greatest cancellative homomorphic image is isomorphic to M. M_∞ is a maximal subsemigroup of S with this property.*

Proof. It is clear that each M_λ $(1 \leqq \lambda \leqq \infty)$ contains M. Let φ: $S \to S/\eta_a$ be the greatest cancellative homomorphism. By Lemma 4.6 or 4.7, $a \in \mathrm{Piv}(S)$ implies $a \in \mathrm{Piv}(M)$, $a \in \mathrm{Piv}(M_i)$ hence $a \in \mathrm{Piv}(M_\infty)$. Then $\varphi|M_i$ and $\varphi|M_\infty$ are the greatest cancellative homomorphisms. We note that $\varphi(x) = \Psi x$, for all $x \in M$, by Proposition 3.1. By definition, if $z \in M_i$, then $a^i z = a^m b$, for some i, m with $i \leqq m$, and some $b \in M$, hence $\varphi(z) = \varphi(a^{m-i}b) = \Psi(a^{m-i}b) \in \Psi M$. Therefore $\varphi(M_i) \subseteqq \Psi M$. If $z \in M_\infty$, $z \in M_i$ for some i and $\varphi(z) \in \Psi M$ as above. Thus $\varphi(M_\infty) \subseteqq \Psi M$. Suppose $M_\infty \subseteq M'$ and $\varphi(M') \subseteqq \Psi M$. If $z \in M'$, $\varphi(z) = \Psi(c) = \varphi(c) \in M$. Then $a^j z = a^j c$ for some $c \in M$ and $j \in Z_+$.
Hence $z \in M_j \subseteq M_\infty$, i.e., $M' = M_\infty$ and M_∞ is maximal.

Analogously to the operator deriving M_∞ from M defined by (4.9), we define an operator $\mathfrak{Tw}_S$ to any subsemigroup of S.

Definition 4.12. Let S be an SAIF-semigroup of the first kind, and $a \in \mathrm{Piv}(S)$. Let A be a subsemigroup of S containing a. Define $\mathfrak{Tw}_S(A) = \{x \in S;\ a^n \mid a^i x$ in A for some $m, i \in Z_+$ with $m \geqq i\}$. $\mathfrak{Tw}_S(A)$ is called the twig hull of A in S.

$\mathfrak{Tw}_S(A)$ is a subsemigroup of S containing A. By Theorem 4.11, S is twigy if and only if $S = \mathfrak{Tw}_S(M) = M_\infty$. We can show $\mathfrak{Tw}_S(M_\infty) = M_\infty$ for any SAIF-semigroup S of the first kind.

Theorem 4.13. *Let* S *be a commutative semigroup.* S *is an* SAIF-*semigroup of the first kind if and only if* S *is a branch-growth of an* $\overline{\mathfrak{N}}$-*semigroup* M *with respect to* a *of* Piv(M), *equivalently,* S *is a branch-growth of a twigy semigroup* T *with respect to* a *of* Piv(T).

Deal with $\mathfrak{M}_S(M)$ of M for the balance of this paper. Let S be an SAIF-semigroup of the first kind, $a \in \mathrm{Piv}(S)$, and M an $\overline{\mathfrak{N}}$-maximal subsemigroup of S containing a. $M_i\,(0 \leqq i < \infty)$ were defined before.

In the following lemmas, a^0X (a^0x) denotes X (x), and $M_0 = M$; $m, n, k, \ell \in Z_+^o$.

Lemma 4.14.

(4.14.1) *If* $m \geqq k \geqq 0$, $n \geqq \ell \geqq 0$ *and* $m + \ell \geqq n + k$, *then* a^mM_n *is an ideal of* a^kM_ℓ.

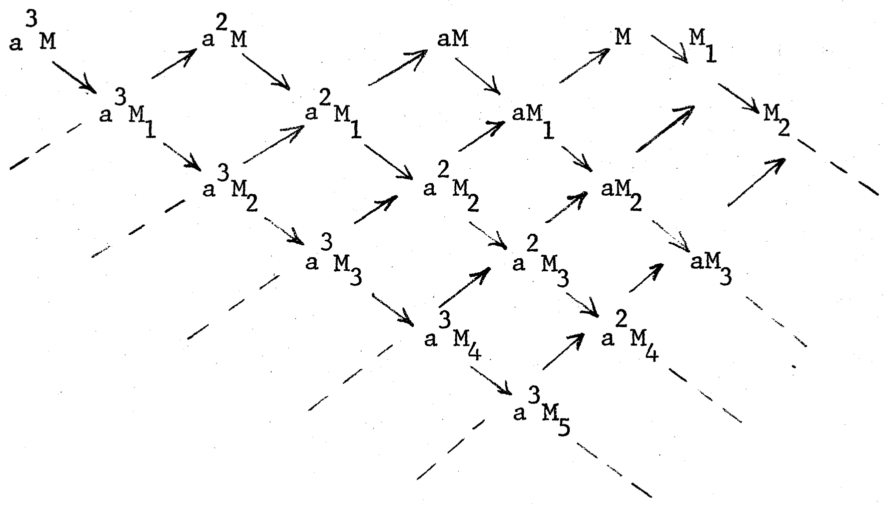
(4.14.2) *If* $n \geqq \ell$, *then* $a^mM_\ell \subseteqq a^mM_n$ *for each* $m \in Z_+^o$.

Proof of (4.14.1). Let $x \in M_n$, $a^nx = a^ib$, $i \geqq n$, $b \in M$. To prove $a^k \mid a^mx$ in M_ℓ, we show $a^{m-k}x \in M_\ell$. As $\ell + m - k \geqq n$ by assumption, $a^\ell a^{m-k}x = a^{\ell+m-k-n}a^nx = a^{\ell+m-k-n+i}b$.
Since $m - k \geqq 0$ and $i - n \geqq 0$, $\ell + m - k - n + i \geqq \ell$. Hence $a^{m-k}x \in M_\ell$. Thus $a^mM_n \subseteqq a^kM_\ell$. Next we show that a^mM_n is an ideal of a^kM_ℓ. As $M_\ell \subseteqq M_n$,

$$a^mM_n \cdot a^kM_\ell = a^{m+k}M_n \cdot M_\ell \subseteqq a^{m+k}M_n \subseteqq a^mM_n .$$

Proof of (4.14.2) By Proposition 4.10, $M_\ell \subseteqq M_n$ if $n \geqq \ell$. Hence $a^mM_\ell \subseteqq a^mM_n$.

We have the following commutative diagram where the arrow $\rightarrow$ shows inclusion.



As M is cancellative, a^mM_n is cancellative if $m \geqq n$. Let $S_0 = \mathrm{Piv}(S)$. Recall $a \in S_0$. Let A_n be the archimedean component of M_n such that $a \in A_n$, $0 \leqq n < \infty$, that is, $A_n = \mathrm{Piv}(M_n)$.

<u>Lemma</u> 4.15. <u>If</u> $m \geqq n \geqq 0$ <u>but</u> $m > 0$, <u>then</u> a^mM_n <u>is an</u> $\mathfrak{N}$-<u>subsemigroup and an ideal of</u> A_n.

Proof. Since $M_n \subseteqq M_{n+1}$, we get $A_n \subseteqq A_{n+1}$ by Lemma 4.8. If $x \in M_n$, $a^mx \in M$, $m \geqq n$, by definition, but $a \in A_n$ implies $a^mM_n \subseteqq A_nM_n \subseteqq A_n$, and $a^mM_nA_n \subseteqq a^mM_n^2 \subseteqq a^mM_n$, hence a^mM_n is an ideal of A_n. As A_n is an AIF-semigroup, a^mM_n is also. By Lemma 4.14, $a^mM_n \subseteqq M$ if $m \geqq n$, hence a^mM_n is cancellative; thus a^mM_n is an $\mathfrak{N}$-semigroup.

Since $a \in M_m$, a^mM_m is an ideal of M_m for each $m \in Z_+$. Also a^mM_m is an $\mathfrak{N}$-subsemigroup of A_m for each $m \in Z_+$ by Lemma 4.15; $a^{m+1}M_{m+1}$ is an ideal of a^mM_m for each $m \in Z_+$ by Lemma 4.14. As aM_1 is archimedean,

$$aM_1 \subseteqq A_0 = \mathrm{Piv}(M).$$

$$\begin{tikzcd}
M \arrow[r] & M_1 \arrow[r] & M_2 \arrow[r] & M_3 \arrow[r] & \cdots \\
A_0 \arrow[u] \arrow[r] & A_1 \arrow[u] \arrow[r] & A_2 \arrow[u] \arrow[r] & A_3 \arrow[u] \arrow[r] & \cdots \\
 & aM_1 \arrow[ul] \arrow[u] & a^2M_2 \arrow[u] \arrow[l] & a^3M_3 \arrow[u] \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Definition 4.16. Let S be an SAIF-semigroup. S is called essential if S contains a cancellative ideal. This definition is due to [4].

Theorem 4.17. Let S be a commutative semigroup. S is a twigy semigroup if and only if S is the union $\bigcup_{i=0}^{\infty} T_i$ of essential twigy semigroups T_i such that

$$T_0 \subseteq T_1 \subseteq \cdots \subseteq T_i \subseteq \cdots ,$$

T_0 is an $\overline{\mathfrak{N}}$-semigroup and $\mathfrak{M}_{T_i}(T_0) = T_i$ for each $i \in Z_+$.

Proof. Necessity is already proved by Theorem 4.11 and Lemma 4.15. Sufficiency. It follows from Lemma 4.7 that S is an SAIF-semigroup of the first kind. Also we see $\mathrm{Piv}(T_0) \subseteq \mathrm{Piv}(S)$ since S is a branch-growth of T_0. It can be shown that for each $x \in S$, $x \,\eta_a\, b$ for some $b \in T_0$ and $a \in \mathrm{Piv}(T_0)$, hence $S/\eta_a \cong T_0$. Thus S is twigy.

We can state the process of construction of a twigy semigroup T as follows: Let M be an $\overline{\mathfrak{N}}$-semigroup. Let $I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$ be a sequence of $\mathfrak{N}$-semigroups such that $I_0 = \mathrm{Piv}(M)$ and I_i is an ideal of I_{i-1} for each $i \in Z_+$. Let $T_0 = M$, and T_1 be a commutative ideal extension of I_1 by a commutative nil-semigroup N_1 such that $T_0 \subset T_1$. Next, let T_2 be a commutative ideal extension of I_2 by a commutative nil-semigroup N_2 such that $T_1 \subseteq T_2$. Continue this process and obtain $T = \bigcup_{i=0}^{\infty} T_i$. As I_i is an $\mathfrak{N}$-semigroup ($i \in Z_+$), the ideal extensions are not difficult.

In this paper we do not discuss precisely how to construct T or T_i ($i \in Z_+$) and how to get an SAIF-semigroup S of the first kind from T as a branch-growth.

Let S be an SAIF-semigroup, $a \in \mathrm{Piv}(S)$, and let M be a maximal $\overline{\mathfrak{N}}$-subsemigroup of S containing a. For each $i \in Z_+$, define

$$M^{(1)} = \{x \in S\colon a\,x \in M\},$$

$$M^{(i)} = \{x \in S\colon a^m | a^i x \text{ in } M \text{ for some } m < i\},\ i > 1.$$

$$M^{(\infty)} = \bigcup_{i=1}^{\infty} M^{(i)}.$$

Then we get the following commutative diagram where $\rightarrow$ shows inclusion.

$$\begin{array}{ccccccccccccc} M & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & \cdots & \longrightarrow & M_i & \longrightarrow & \cdots & \longrightarrow & M_\infty \\ & & \downarrow & & \downarrow & & & & \downarrow & & & & \downarrow \\ & & M^{(1)} & \longrightarrow & M^{(2)} & \longrightarrow & \cdots & \longrightarrow & M^{(i)} & \longrightarrow & \cdots & \longrightarrow & M^{(\infty)} \end{array}$$

We can see $S = M^{(\infty)}$.

Proposition 4.18. *Let S be an SAIF-semigroup of the first kind. S is essential (and twigy) if and only if there is an $a \in \mathrm{Piv}(S)$ and a maximal cancellative subsemigroup M containing a such that $S = M^{(i)}$ $(S = M_i)$ for some $i \in Z_+$.*

However the two concepts, essentiality and twiginess, are independent as the following two examples show.

Example 4.19. Here is an example of an archimedean twigy semigroup which is not essential. The proof is left for the reader's exercise.

Le $S' = \{(m,x)\colon m,\ x \in Z_+^o\}$ and define

$$(m,x)\ (n,y) = (\max\{m,n\},\ x+y+1).$$

Define ρ by

$$(4.19.1)\qquad (m,x)\rho(n,y) \quad \text{iff} \begin{cases} \text{either } m = n \text{ and } x = y \\ \\ \text{or } \quad m \neq n \text{ and } x = y \geqq \mathrm{Max}\{m,n\}. \end{cases}$$

Then S' is a semigroup and ρ is a congruence on S'. Let $S = S'/\rho$. We can show that S is an AIF-semigroup. For notational convenience, any element of S is still denoted by (m,x), and then (4.19.1) is regarded as the definition of

$(m,x) = (n,y)$. Let $M = \{(0,x): x \in Z^o_+\}$. $M \cong Z_+$ and M is a maximal cancellative subsemigroup of S. The projection $(m,x) \to x$ is the greatest cancellative homomorphism of S as shown in the following: First $(m,x)\eta_{(0,0)}(n,y)$ implies $x = y$. Conversely if (m,x) and (n,x) are given and if $k \geqq \max\{m,n\}$, then we get

$$(0,0)^k(m,x) = (0,0)^k(n,x).$$

Hence S is twigy. Suppose S has a cancellative ideal J. Let $a = (0,0)$. As S is archimedean, for $b \in J$, there is $m \in Z_+$ and $c \in S$ such that $a^m = bc$. Hence $a^m \in J$, and $a^mS^1 \subseteq J$; thus a^mS^1 is cancellative for some $m \in Z_+$. However we show a^mS^1 is never cancellative as follows:

Let $n > m$. Then

$$(0,0)^m(n,0) = (0,m-1)(n,0) = (n,m),$$

$$(0,0)^m(0,0) = (0,m-1)(0,0) = (0,m),$$

and $(n,m) \neq (0,m)$ since $m < n$.

On the other hand,

$$(0,0)^{n-m}(n,m) = (0,n-m-1)(n,m) = (n,n),$$

$$(0,0)^{n-m}(0,m) = (0,n-m-1)(0,m) = (0,n),$$

and $(n,n) = (0,n)$. This completes the proof.

Example 4.20. We exhibit an example of an essential SAIF-semigroup which is not twigy.

Let $A = \{(a,x): a \in Z,\ x \in Z_+\}$, $B = \{[b,m]: b \in Z,\ m \in Z_+\}$. Considering the set Z_+ and a letter $\mathbf{0}$, define

$$S = A \cup \{\mathbf{0}\} \cup B \cup Z_+$$

and define the commutative binary operation in S by

$$(a,x)\cdot(b,y) = (a+b, x+y+1) \quad \text{for } a, b \in Z,\ x, y \in Z_+$$

$$\mathbf{0}\cdot(b,y) = (b,y+1) \quad \text{for } b \in Z,\ y \in Z_+$$

$$\mathbf{0}\cdot\mathbf{0} = (0,1)$$

$$[a,m]\cdot[b,n] = [a+b,m+n+1] \qquad \text{for } a,\ b \in Z,\ m,\ n \in Z_+$$

$$a\cdot[b,n] = [a+b,n+1] \qquad \text{for } a \in Z_+,\ b \in Z,\ n \in Z_+$$

$$a\cdot b = [a+b,1] \qquad \text{for } a,\ b \in Z_+$$

$$(a,x)\cdot[b,n] = (a+b,x+n+1) \qquad \text{for } a,\ b \in Z,\ m,\ n \in Z_+$$

$$\mathbf{0}\cdot[b,n] = (b,n+1) \qquad \text{for } b \in Z,\ n \in Z_+$$

$$a\cdot\mathbf{0} = (a,1) \qquad \text{for } a \in Z_+$$

$$a\cdot(b,y) = (a+b,y+1) \qquad \text{for } a \in Z_+,\ b \in Z,\ y \in Z_+.$$

We can easily show associativity of S since S is isomorphic into $Z \times Z_+ \times L = \{((a,z,i)) : a \in Z,\ z \in Z_+,\ i \in L\}$, the direct product of the group Z, the semigroup Z_+ and the semilattice L of order 2, $L = \{0,1\}$, $0^2 = 0\cdot 1 = 1\cdot 0 = 0$, $1^2 = 1$. In fact the isomorphism $S \to Z \times Z_+ \times L$ is given by

$$(a,x) \longrightarrow ((a,x+1,0))$$

$$\mathbf{0} \longrightarrow ((0,\ 1,\ 0))$$

$$[a,m] \longrightarrow ((a,m+1,1))$$

$$a \longrightarrow ((a,\ 1,\ 1)).$$

Both $A \cup \{\mathbf{0}\}$ and $B \cup Z_+$ are AIF-semigroups and $A \cup \{\mathbf{0}\}$ is an ideal of S. Moreover, A is a cancellative ideal of S, and therefore S is an essential SAIF-semigroup.

Consider the smallest cancellative congruence $\eta_{\mathbf{0}}$ on S. Note that $\mathbf{0}$ is a pivot element of S. Let $X_0 \in Z_+ \cup \{\mathbf{0}\}$ and $Y \in S$. One can show that if $\mathbf{0}^k\cdot X_0 = \mathbf{0}^k\cdot Y$ for some $k \in Z_+$ then $X_0 = Y$. This shows that $\{\mathbf{0}\}$ and $\{a\}$ are $\eta_{\mathbf{0}}$-classes for all $a \in Z_+$. Suppose S is twigy. Then S must have a cancellative subsemigroup M which contains $\mathbf{0}$ and Z_+. Let $a \in M \cap Z_+$. Then $a\cdot a = [2a,1] \in M$. Of course $2a \in M$ by the above remark; moreover $(2a,1) \in M$ since $\mathbf{0}\cdot 2a = (2a,1)$. However

$$\mathbf{0}\cdot(2a,1) = (2a,2) = \mathbf{0}\cdot[2a,1], \quad (2a,1) \neq [2a,1].$$

This is a contradiction. Therefore S is not twigy.

The following are unsolved:

Let B be an SAIF-semigroup of the first kind. Given B and $a \in B$. Construct branch-growths A of B with respect to a. Especially, if M is an $\overline{\mathfrak{N}}$-maximal subsemigroup and $a \in M$, how can we construct $M^{(\infty)}$ from M_∞?

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